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Estimation of predictive uncertainties in flood wave propagation in a river channel using adjoint sensitivity analysis

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SUMMARY

This paper applies adjoint sensitivity analysis to flash flood wave propagation in a river channel. A numerical model, based on the St-Venant equations and the corresponding adjoint equations, determines the sensitivities of predicted water levels to uncertainties in key controls such as inflow hydrograph, channel topography, frictional resistance and infiltration rate. Sensitivities are calculated in terms of a measuring function that quantifies water levels greater than certain safe threshold levels along the channel. The adjoint model has been verified by means of an identical twin experiment. The method is applied to a simulated flash flood in a river channel. The sensitivities to key controls are evaluated and ranked and the effects of individual and combined uncertainties on the predicted flood impact are also quantified.

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KEY WORDS: data assimilation; open channel flow; adjoint sensitivity analysis; numerical models; St-Venant equations

1. INTRODUCTION

The quality of flood predictions by numerical models depends on the accuracy of the inflow hydrograph and other control variables such as bed roughness, bed slope and infiltration rate. Each of these has its own effect on predicted flood levels. This paper examines the effects of uncertainties in the values of these controls on the flood prediction. That is, we find out how the overall sensitivity of the predicted flood level can be apportioned to the individual sensitivities to each of the control variables. This could be done at significant computational expense using multiple runs and ensemble techniques. However, the adjoint method presented here determines these sensitivities in one run of the model. Flood wave propagation models are often used when planning flood management strategies and it is important to consider what control actions could

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mitigate flood impact. Such controls could be hydraulic structures such as gates, locks, and weirs or the diversion of water into canals and floodplain storage facilities [1] or abstraction for flood control [2]. However, the term 'control' is also used for a user-defined value that determines the result of a model forecast. Sensitivities to such controls can also be used to identify distributed coefficients such as Manning's roughness [3], initial or boundary conditions [4, 5], for ranking the parameters according to their effect on the flood level [6] and to quantify the uncertainty in the predicted flood level to uncertainties in control variables as shown in this paper. The novelty of this paper is in applying the adjoint sensitivities for uncertainty analysis to the model governed by the St-Venant equations (StVEs).

2. GOVERNING EQUATIONS FOR OPEN CHANNEL FLOW

The StVEs that include the effect of infiltration rate into the bed are

$$\begin{gathered}\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}-fb=0\\ \frac{\partial Q}{\partial t}+gA\left(\frac{\partial H}{\partial x}+\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial x}(Qu)+\frac{kQ|Q|}{AR}+\left(\frac{uf}{2}\right)b=0\end{gathered} \tag{1}$$

where t is time, x is the distance along the channel, Q is the discharge, A is the flow cross-section area, H is the total water stage, g is the gravitational acceleration, z is the vertical distance between the horizontal datum and the channel bed, $k=gn^2/R^{1/3}$ is a friction factor (Manning), $(\partial/\partial x)(Qu)$ is the momentum flux term, b is the channel bottom width and f is the infiltration rate. The effect of infiltration rate is added to the StVEs using the Green–Ampt model [7] as follows:

$$f=-K\left(\psi_f+\frac{F}{(\theta_s-\theta_0)}-H_f\right)\frac{(\theta_s-\theta_0)}{F} \tag{2}$$

where F is the cumulative depth of infiltration, K is saturated hydraulic conductivity, ψ_f is suction at the wetting front (negative pressure head), θ_0 is initial moisture content, θ_s is saturated moisture content and H_f is the depth of ponding.

3. ADJOINT SENSITIVITY ANALYSIS FOR StVEs

We define a quadratic measuring function, r, which compares predicted water levels with threshold values, H_d, above which flooding may occur. It is defined in terms of the corresponding threshold wetted cross-sectional area A_d as follows:

$$r=0.25\{(A-A_d)^2+(A-A_d)|A-A_d|\}\delta(x-x_0)\delta(t-t_0) \tag{3}$$

where $A(x,t)$ is the flow cross-section area calculated by the forward hydraulic model, and $A_d(x_0,t_0)$ are threshold values at locations $x=x_0$, and times $t=t_0$. Note that $r=0$ for $A<A_d$

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and $r=0.5(A-A_d)^2$ for $A>A_d$. Following [2], we define the Lagrangian J as follows:

$$J=\int_0^T\int_0^L\left\{r+\phi\left[\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}-fb\right]\right.$$
$$\left.+\psi\left[\frac{\partial Q}{\partial t}+gA\left(\frac{\partial H}{\partial x}+\frac{\partial z}{\partial x}\right)+\frac{\partial}{\partial x}(Qu)+\frac{kQ|Q|}{AR}+\left(\frac{uf}{2}\right)b\right]\right\}\mathrm{d}x\,\mathrm{d}t \tag{4}$$

where weights ϕ and ψ are Lagrange multipliers later to be revealed as the adjoint variables and L and T are the spatial and temporal limits of the domain. Following [3, 6] the adjoint equations are found to be

$$\begin{aligned}\frac{\partial\phi}{\partial\tau}-gH\frac{\partial\psi}{\partial x}+\psi g\frac{\partial z}{\partial x}-\psi gn^2\frac{Q|Q|}{A^2R^{4/3}}+\frac{Q^2}{A^2}\frac{\partial\psi}{\partial x}+\frac{\partial r}{\partial A}+\frac{\partial f}{\partial H}\left(\frac{\psi b}{2}\frac{Q}{A}-b\phi\right)-\frac{\psi b}{2}\frac{Q}{A^2}f=0\\ \frac{\partial\psi}{\partial\tau}-\frac{\partial\phi}{\partial x}+2\psi gn^2\frac{|Q|}{AR^{4/3}}-2\frac{Q}{A}\frac{\partial\psi}{\partial x}+\frac{\partial r}{\partial Q}+\frac{\psi}{2}\frac{bf}{A}=0\end{aligned} \tag{5}$$

where $\tau=(T-t)$. Sensitivity to the inflow $Q(0,t)$ is found to be $\partial J/\partial Q=-\int_0^T\{\phi+2u\psi\}\,\mathrm{d}t$, the sensitivity to the bed slope $S_0=-\partial Z/\partial x$ is found to be $\partial J/\partial S_0=-\int_0^T\int_0^L\{g\psi A\}\,\mathrm{d}x\,\mathrm{d}t$, while the sensitivity to the bed friction in terms of Manning's coefficient, n, is found to be $\partial J/\partial n=\int_0^T\int_0^L\{2\psi gn(Q|Q|/AR^{4/3})\}\,\mathrm{d}x\,\mathrm{d}t$ and the sensitivity to the infiltration rate is found to be $\partial J/\partial f=\int_0^T\int_0^L\{(u/2)\psi-\phi\}\,\mathrm{d}x\,\mathrm{d}t$. These sensitivities can be evaluated after the adjoint equations (5) are solved for ϕ and ψ.

4. EVALUATING THE UNCERTAINTIES FROM THE ADJOINT SENSITIVITIES

A simple space and time staggered finite difference mesh is used to discretize the domain for both the forward and adjoint models. Lateral boundaries are made transparent to outgoing waves using characteristics to interpolate boundary values of A, ϕ and ψ from interior values. The forward model has been validated against certain idealized test cases for both steady and unsteady flow. These have been reported elsewhere [6]. The adjoint model is validated using a classical identical twin experiment based on the conjugate gradient method. Results demonstrate rapid convergence such that the measuring function is reduced by a factor of 10^5 in about 10 iterations. The calculated sensitivities have also been verified by repeating a test case published in [1] and achieving the same numerical results.

The uncertainties in each control variable are assumed to be distributed as a Gaussian probability density function and so the $\mathrm{Var}(J)$ can be used as a measure of uncertainty:

$$\mathrm{Var}(J)=\left(\frac{\partial J}{\partial Q}\right)^2\mathrm{Cov}(Q)+\left(\frac{\partial J}{\partial f}\right)^2\mathrm{Cov}(f)+\left(\frac{\partial J}{\partial n}\right)^2\mathrm{Cov}(n)+\left(\frac{\partial J}{\partial S_0}\right)^2\mathrm{Cov}(S_0) \tag{6}$$

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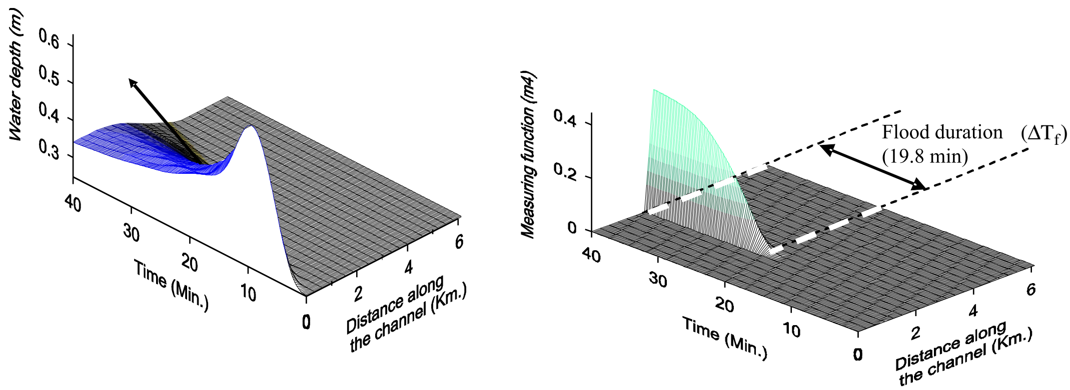

Figure 1. Left: flash flood wave propagation; right: measuring function r as defined in (3).

where $\partial J/\partial Q$, $\partial J/\partial A$, $\partial J/\partial f$, $\partial J/\partial n$ and $\partial J/\partial S_0$, are the sensitivities of flood levels at $x=x_0$ to uncertainties δQ, δA, δf, δn and δS_0 in each of these control variables assuming no correlation between the control variables.

However, when considering the discretized model we should use a matrix formulation. For example, if we have uncertainty in one control variable, Manning's coefficient, n, then the total variance in the objective function, J, due to the covariance of n is given by

$$\text{Var}(J)=\left[\frac{\partial J}{\partial n}\right]^{\text{T}}[\text{Cov}(n)]\left[\frac{\partial J}{\partial n}\right] \tag{7}$$

where $\partial J/\partial n = 2\psi g n(Q|Q|/AR^{4/3})$ is the vector of local sensitivities computed in each cell and $\text{Cov}(n)$ is the covariance matrix. As engineers, we need a physical interpretation of the objective function and its variance. The objective function, $J=\int\int r\,dx\,dt$, unit ($\text{m}^5\,\text{s}$), is an integral measure of flood impact. Now, since $r=0$ for $A<A_d$ and $r=0.5(A-A_d)^2$ for $A>A_d$, we can scale J by the mean flood excess cross-sectional area defined by $I=(0.25/LT)\int\int\{(A-A_d)+|A-A_d|\}\delta(x-x_0)\delta(t-t_0)\,dx\,dt$. Consequently, we can define the flood impacts, $P=J/I\,(\text{m}^3\,\text{s})$ and $\text{Var}(P)=\text{Var}(J)/I^2(\text{m}^3\,\text{s})^2$, which have more physically convenient units. We are more likely to be interested in the flood impact in a particular reach rather than over the whole domain. Hence, it is useful to note that we are free to decide where to evaluate the flood impact, P. If, for example, we are interested in the flood impact for a reach of length ΔL_f and for duration of the flood event ΔT_f as shown in Figure 1 (right), then these length and time scales can be used to scale J and I. This is achieved by controlling the non-zero part of the kernel of the integrals that define J by means of the Kronecker delta functions $\delta(x-x_0)$ and $\delta(t-t_0)$, where locations $x_0\in\Delta L_f$ and times $t_0\in\Delta T_f$.

5. APPLICATION

5.1. Channel description and parameter uncertainties

The method was applied to a simulated flash flood in a river channel of trapezoidal section with an average bottom width $b=14\,\text{m}$ and (1:1) side slope. A sinusoidal hydrograph of duration 13 min

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and peak discharge $9.4\,\mathrm{m^3\,s^{-1}}$ is introduced at the upstream boundary, while the total simulation period is $T = 40\,\mathrm{min}$. The channel length is $L = 6\,\mathrm{km}$. Measuring stations are located 2 km from the upstream boundary. The Manning friction coefficient is $n = 0.016\,\mathrm{m^{-1/3}\,s}$, the uniform bed slope, $S_0 = 0.027$. The infiltration rate, f in (2), is a function of $K = 10^{-5}\,\mathrm{m\,s^{-1}}$; $\psi_f = -0.10\,\mathrm{m}$; $\theta_0 = 0.318$; and $\theta_s = 0.518$ [8].

We assume that the variance of each control variable is uniform along the channel and has the following values; $\mathrm{Var}(n) = 10^{-5}\ (\mathrm{m^{-1/3}\,s})^2$, $\mathrm{Var}(S_0) = 10^{-7}$ and $\mathrm{Var}(f) = 10^{-10}\ (\mathrm{m\,s^{-1}})^2$. The variance in Manning coefficient n follows [9], and represents an uncertainty in n of approximately 20%. The variance in S_0 represents an uncertainty in the bed slope of approximately 1%. The variance in the infiltration rate, f, which is spatially and temporarily variable and difficult to measure, represents an uncertainty of 50%. The variance in the upstream discharge is assumed to be proportional to Q^2 such that $\mathrm{Var}(Q) = 0.001\,Q^2$. A correlation function for each control variable is assumed to have a correlation radius of two model cells ($2\Delta x$) and allows the covariance matrix to be defined.

5.2. *Flood event and flood impact*

The flood wave is illustrated in Figure 1 (left), which shows computed water depths plotted against time and distance along the channel. The effect of the inflow hydrograph at $x = 0$ is clearly shown from 0 to 13 min, while the effect of channel storage is evident up to $t = 40\,\mathrm{min}$. The wave propagates and decays with distance along the channel. The leading edge of the wave front is shown by the arrow. The measuring locations $x_0 \in \Delta L_f$ are located 2 km from the upstream boundary. A threshold cross-sectional area $A_d = 5.0\,\mathrm{m^2}$ is chosen at $x_0 = 2.0\,\mathrm{km}$ with $\Delta L_f = 1.0\,\mathrm{m}$. This represents a threshold water level of 0.26 m above the bed. A small value such as this indicates only the presence of the flood wave rather than a significant risk of damage or danger. Once the forward model has been integrated, the objective function, J, is evaluated and found to be $322.4\,\mathrm{m^5\,s}$, while the scaling function is $I = 0.34\,\mathrm{m^2}$. The flood duration, i.e. when $A > A_d$, $\Delta T_f = 19.8\,\mathrm{min}$ as shown in Figure 1 (right), shows the measuring function r; consequently the flood impact, $P = 952.4\,\mathrm{m^3\,s}$. This is equivalent to a mean excess wetted cross-sectional area (i.e. wetted cross-sectional area greater than $A_d = 5.0\,\mathrm{m^2}$) equal to $(952.4)/(\Delta T_f)(\Delta L_f) = 0.8\,\mathrm{m^2}$.

5.3. *Adjoint sensitivities and uncertainties*

The adjoint equations (5) are solved to evaluate the adjoint variables (ϕ) and (ψ), which are then used to quantify the sensitivities defined in Section 3. As examples, Figure 2 (left) and (right) shows ($\partial J/\partial S_0$), and ($\partial J/\partial n$), respectively. Non-zero sensitivities can be seen to follow the trajectory of the flood wave but propagate in reverse time, away from (ΔL_f) and (ΔT_f) toward the upstream boundary. In Section 5.2 we evaluate the total flood impact, P, that arises from a water depth excess above H_d within (ΔL_f) and (ΔT_f). We now need estimates of the uncertainty in this flood impact. This is evaluated through the variance in the objective function, $\mathrm{Var}(J)$, and the variance in the flood impact, $\mathrm{Var}(P)$. We can also represent uncertainty by the standard deviation of the flood impact, $\sigma_p = [\mathrm{Var}(P)]^{1/2}$. Values for these measures of uncertainty for each control variable are given in Table I. These results, based on plausible levels of uncertainty in each control variable, indicate that in this particular example the predicted flood impact is most sensitive to the upstream discharge and least sensitive to the infiltration rate. We also compute a percentage uncertainty in the whole flood impact $(\sigma_p/P)*100$, which is about 5.8%. These results

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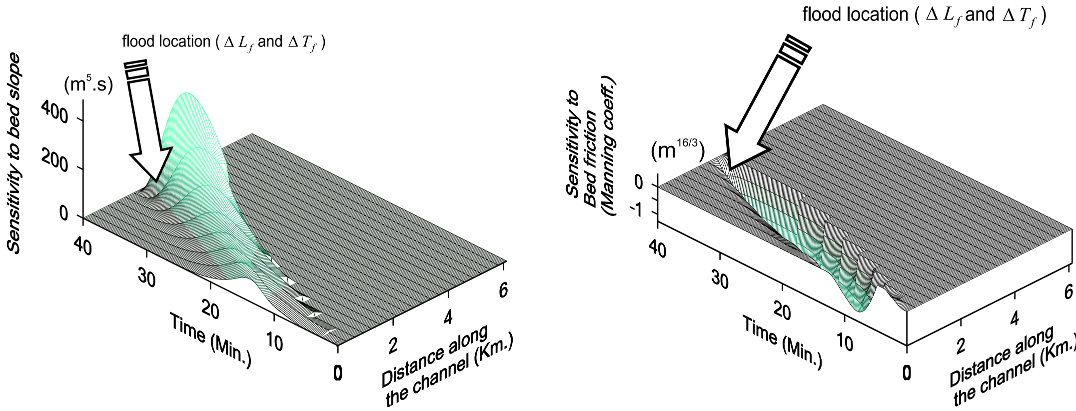

Figure 2. Left: sensitivity to bed slope; right: sensitivity to bed friction (Manning coefficient).

Table I. Computed measures of uncertainty in the flood impact due to uncertainties in the controls.

Control	Variance in J $(\mathrm{m}^5\,\mathrm{s})^2$	Variance in the flood impact $(\mathrm{m}^3\,\mathrm{s})^2$	Standard deviation in the flood impact $(\mathrm{m}^3\,\mathrm{s})$
Due to bed slope (S_0)	44.4	388	19.7
Due to Manning coefficient (n)	0.60	5.25	2.29
Due to infiltration (f)	6.15E−4	5.36E−3	7.33E−2
Due to upstream discharge (Q)	294	2654	50.5
Total	339	3047	74.5

demonstrate both the ranking of controls in any given flood event and the combined predictive uncertainties.

6. CONCLUSIONS

The adjoint method is applied to find the sensitivity of flood impact to uncertainties in some control variables. A flood event is simulated by solving the StVEs and sensitivity information is found from the solution of the adjoint equations. This allows the overall uncertainty in flood impact to be calculated in a single model run. This is more efficient than the conventional methods based on ensemble techniques and reveals much more about the propagation of uncertainty through the model domain.

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